ISRAEL JOURNAL OF MATHEMATICS 89 (1995), 141–155

COMPLETIONS OF COMMUTATIVE RINGS AND MODULES

BY

PASCUAL JARA AND EVA SANTOS*

Department of Algebra
University of Granada
18071 Granada, Spain

ABSTRACT

In this work we continue studying the notion of completion of R-modules, over a commutative ring R, relative to a torsion theory σ. We develop some techniques relative to localization at prime ideals and give structural results on the completion of finitely generated R-modules, describing it as the product of classical completions on local noetherian rings.

Introduction

A fundamental tool in the study of the Matlis duality for a commutative local ring $(R, \mathfrak{m})$ is the completion with respect to the maximal ideal $\mathfrak{m}$, cf. [13]. This theory had been extended to a more general setting in [4], where the completion of a ring R was defined in terms of torsion theories, thus yielding back the aforementioned Matlis duality theory as an special case.

Later in [3] the authors introduced and studied the general definition of completion of an arbitrary module, with respect to a torsion theory, and established its first properties.

Continuing the study of completion, some structure results are necessary to complete the theory. One of the aims of this paper is to show that the completion

* The authors acknowledge partial support from the D.G.I.C.Y T.
Received May 25, 1993

of a module can be expressed, in an elegant way, as a direct product of classical completions on local rings obtained by localizing at prime ideals. For the ring R, this result follows easily from [4]. That simple proof can not be adapted to σ-finitely generated R-modules, however.

We begin by collecting some introductory remarks about the subject of completion; most of the results can be found in either [3] or [4], and we refer to [6] and [15] for undefined terms relating to torsion theories.

Let R be a unitary commutative ring and let σ be a torsion theory in the category $R-\text{mod}$ of unitary R-modules. We say that an R-module M is σ-noetherian if the set

$$\mathcal{C}_\sigma(M) = \{N \subseteq M: M/N \in \mathcal{F}_\sigma\}$$

satisfies the ascending chain condition with respect to the inclusion. The ring R is said to be σ-noetherian if it is σ-noetherian as an R-module. In the literature there are many different characterizations of σ-noetherian rings and R-modules; we refer to [6] or [11] for a rather complete sources on this subject.

Let us introduce two useful examples of torsion theories in the category R-mod. Let $\mathfrak{p}$ be a prime ideal of R, the torsion theory cogenerated by $R/\mathfrak{p}$, see [6], has as Gabriel filter the set of all ideals I such that $I \not\subseteq \mathfrak{p}$, it coincides with the torsion theory determined by the multiplicative set $R \setminus \mathfrak{p}$, hence we denote by $\sigma_{R\setminus\mathfrak{p}}$ this torsion theory. If in addition R is a noetherian ring or I is a finitely generated ideal of R, the torsion theory σ_I generated by R/I, see [6], has an easy description, thus its Gabriel filter consists of those ideals J such that $I^n \subseteq J$ for some positive integer n.

If σ is a torsion theory in $R-\text{mod}$, then there are several sets of prime ideals which determine the torsion theory σ, at least when R is σ-noetherian. They are:

$$\mathcal{Z}(\sigma) = \{\mathfrak{p} \in \text{Spec}(R): R/\mathfrak{p} \in \mathcal{T}_\sigma\}$$

as well as

$$\mathcal{K}(\sigma) = \{\mathfrak{p} \in \text{Spec}(R): R/\mathfrak{p} \in \mathcal{F}_\sigma\}$$

and $\mathcal{C}(\sigma)$, the set of all maximal elements in $\mathcal{K}(\sigma)$. The sets $\mathcal{K}(\sigma)$ and $\mathcal{Z}(\sigma)$ form a partition of Spec(R). Moreover, $\mathcal{K}(\sigma)$ is **generically closed** in Spec(R), with the Zariski topology, (i.e., if $\mathfrak{p}$, $\mathfrak{q} \in \text{Spec}(R)$ satisfy $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{K}(\sigma)$, then $\mathfrak{q} \in \mathcal{K}(\sigma)$) and $\mathcal{Z}(\sigma)$ is **closed under specialization**, (i.e., if $\mathfrak{p}$, $\mathfrak{q} \in \text{Spec}(R)$

satisfy $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \mathcal{Z}(\sigma)$, then $\mathfrak{p} \in \mathcal{Z}(\sigma)$). If the ring R is noetherian then the torsion theory σ can be expressed as

$$\sigma = \bigvee\{\sigma_{\mathfrak{p}}: \mathfrak{p} \in \mathcal{Z}(\sigma)\} = \bigwedge\{\sigma_{R\smallsetminus\mathfrak{p}}: \mathfrak{p} \in \mathcal{K}(\sigma)\} = \bigwedge\{\sigma_{R\smallsetminus\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\sigma)\}.$$

In more general situations, this may fail to hold. Yet, if R is a σ-noetherian ring, then the torsion theories τ bigger that σ are still determined by $\mathcal{K}(\tau)$. In this case we have:

$$\tau = \bigwedge\{\sigma_{R\smallsetminus\mathfrak{p}}: \mathfrak{p} \in \mathcal{K}(\tau)\} = \bigwedge\{\sigma_{R\smallsetminus\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\tau)\}.$$

There exists another property shared by all torsion theories in a noetherian ring: stability. We say that a torsion theory σ is **stable** if the class $\mathcal{T}_\sigma$ is closed under taking essential extensions. As a consequence of the Artin-Rees Lemma, every torsion theory on a noetherian ring is stable. In the relative case we can prove a similar result, i.e., if σ is an stable torsion theory in $R-\text{mod}$ and R is σ-noetherian, then every torsion theory $\tau \geq \sigma$ is also stable, see [5, Lemma VI.3.5].

With this background, we may start properly our theory. Let σ be a stable torsion theory in $R-\text{mod}$ such that R is σ-noetherian; following [5, page 157], we define the first skeleton of σ as the torsion theory σ^1 such that $\mathcal{Z}(\sigma^1) = \mathcal{Z}(\sigma) \cup \mathcal{C}(\sigma)$. So $\sigma \leq \sigma^1$, hence σ^1 is stable and R is σ^1-noetherian by the above remarks.

Let us include at this point some easy examples. Let σ be the trivial torsion theory on a noetherian local ring $(R, \mathfrak{m})$, i. e., $\mathcal{T}_\sigma = \{0\}$, then σ^1 is the torsion theory generated by the simple R-module $R/\mathfrak{m}$. Another typical example is the following: let R be a Krull domain and σ the torsion theory such that $\mathcal{K}(\sigma) = \text{Spec}^{(1)}(R) = \{\mathfrak{p} \in \text{Spec}(R): \text{height}(\mathfrak{p}) \leq 1\}$. It is well known that R is σ-noetherian but, in general, not noetherian, cf. [15]. So the theory we will develop can be applied to this particularly interesting example, as well.

If $\sigma \leq \tau$ are torsion theories in $R-\text{mod}$, where R is σ-noetherian and σ is stable, then for any R-module M, we have defined in [3] the (σ,τ)-completion of M by

$$M^{(\sigma,\tau)} = \varprojlim\{Q_\sigma(M/N): M/N \in \mathcal{T}_\tau\},$$

i.e., $M^{(\sigma,\tau)}$ is the completion of $Q_\sigma(M)$ in the category $(R,\sigma)-\text{mod}$ with respect to the torsion theory τ. The reader can consult [8], [10], [12] or [15] as a guide on completion in the case in which σ is the trivial torsion theory.

In particular, if $\sigma \leq \tau \leq \sigma^1$, we obtained additional information on the completion and its related features, i.e., Hausdorff modules, pseudocomplement, etc. It is in this case that we can give a structural result on the completion $M^{(\sigma,\tau)}$. We use the work of E. Matlis on torsion free modules, cf. [9], and the recent paper of W. Brandal, [2], as inspiration, but the methods to prove the main theorem are completely different; they follow mainly the philosophy contained in the paper [4]. Actually, Brandal and Matlis proved a decomposition result only for the ring R, and here we show that it is possible to obtain a similar result for σ-finitely generated R-modules.

1. $(\sigma, \pi_{\mathfrak{p}})$-completion

Throughout this note R will be a commutative ring and σ a stable torsion theory in $R-$ mod, such that R is σ-noetherian.

In what follows let $\mathfrak{p} \in \mathcal{C}(\sigma)$, and denote by $\pi_{\mathfrak{p}}$ the torsion theory determined by the set $\mathcal{Z}(\pi_{\mathfrak{p}}) = \mathcal{Z}(\sigma) \cup \{\mathfrak{p}\}$, i. e., $\pi_{\mathfrak{p}} = \wedge\{\sigma_{R\smallsetminus\mathfrak{q}}\colon \mathfrak{q} \notin \mathcal{Z}(\sigma) \cup \{\mathfrak{p}\}\}$. Observe that $\sigma \leq \pi_{\mathfrak{p}}$, so R is $\pi_{\mathfrak{p}}-$noetherian and $\pi_{\mathfrak{p}}$ is stable.

We are going to compute $R^{(\sigma,\pi_{\mathfrak{p}})}$ step by step.

1. First of all we consider the pseudocomplement $\kappa = (\sigma : \pi_{\mathfrak{p}})$ of $\pi_{\mathfrak{p}}$ relative to σ. It is well known from [3] that $\mathcal{K}(\kappa)$ is the smallest generically closed set containing $\mathcal{K}(\sigma) \cap \mathcal{Z}(\pi_{\mathfrak{p}}) = \mathcal{K}(\sigma) \cap \{\mathfrak{p}\} = \{\mathfrak{p}\}$. Thus $\kappa = \sigma_{R\smallsetminus\mathfrak{p}}$. The localization at this torsion theory will be denoted by $(-)_{\mathfrak{p}}$, as usually.
2. Using [3, Proposition 3.10] and the first step, we thus obtain:

$$R^{(\sigma,\pi_{\mathfrak{p}})} \cong (Q_\kappa(R))^{(\sigma,\pi_{\mathfrak{p}})} = (R_{\mathfrak{p}})^{(\sigma,\pi_{\mathfrak{p}})}.$$

3. We observe that $\pi_{\mathfrak{p}}$ induces a torsion theory $\widetilde{\pi_{\mathfrak{p}}}$ in $R_{\mathfrak{p}}-$ mod by:

$$\text{if } M \in R_{\mathfrak{p}} - \text{mod, then } M \in \mathcal{T}_{\widetilde{\pi_{\mathfrak{p}}}} \text{ if, and only if, } M \in \mathcal{T}_{\pi_{\mathfrak{p}}}.$$

   This torsion theory $\widetilde{\pi_{\mathfrak{p}}}$ yields a partition of $\mathrm{Spec}(R_{\mathfrak{p}})$ consisting of $\mathcal{Z}(\widetilde{\pi_{\mathfrak{p}}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$ and $\mathcal{K}(\widetilde{\pi_{\mathfrak{p}}}) = \mathrm{Spec}(R_{\mathfrak{p}})\smallsetminus\{\mathfrak{p}R_{\mathfrak{p}}\}$. Thus, we have that $\widetilde{\pi_{\mathfrak{p}}}$ is equal to the torsion theory $\sigma_{\mathfrak{p}R_{\mathfrak{p}}}$ generated by the quotient $R_{\mathfrak{p}}$-module $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

In order to obtain the main result of this section, we need the next Proposition:

PROPOSITION 1.1: *If I is an R-submodule of $R_{\mathfrak{p}}$ such that $0 \neq R_{\mathfrak{p}}/I \in \mathcal{F}_\sigma \cap \mathcal{T}_{\pi_{\mathfrak{p}}}$, then $I = I_{\mathfrak{p}}$.*

Proof: Using [5, Lemma III.4.2] and [5, Proposition III.4.13] from the hypothesis we obtain

$$\varnothing \neq \mathrm{Ass}(R_{\mathfrak{p}}/I) \subseteq \mathcal{K}(\sigma) \cap \mathcal{Z}(\pi_{\mathfrak{p}}) = \{\mathfrak{p}\}.$$

Hence the assertion follows from [15, Proposition IX.4.2]. ∎

THEOREM 1.2: *With the same notation as in the above Proposition*

$$(R_{\mathfrak{p}})^{(\sigma,\pi_{\mathfrak{p}})} = \widehat{R_{\mathfrak{p}}},$$

where $\widehat{R_{\mathfrak{p}}}$ is the ordinary $\mathfrak{p}R_{\mathfrak{p}}$-completion of the local ring $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$.

Proof: By definition

$$(R_{\mathfrak{p}})^{(\sigma,\pi_{\mathfrak{p}})} = \varprojlim\{Q_\sigma(R_{\mathfrak{p}}/I)\colon R_{\mathfrak{p}}/I \in \mathcal{F}_\sigma \cap \mathcal{T}_{\pi_{\mathfrak{p}}}\} =$$

and using the previous Proposition we obtain

$$= \varprojlim\{Q_\sigma(R_{\mathfrak{p}}/I_{\mathfrak{p}})\colon R_{\mathfrak{p}}/I_{\mathfrak{p}} \in \mathcal{T}_{\sigma_{\mathfrak{p}R_{\mathfrak{p}}}}\} =$$

now, since the composition $Q_\sigma\ (-)_{\mathfrak{p}}$ is exactly $(-)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{K}(\sigma)$, the equality follows as:

$$= \varprojlim\{R_{\mathfrak{p}}/I_{\mathfrak{p}}\colon R_{\mathfrak{p}}/I_{\mathfrak{p}} \in \mathcal{T}_{\sigma_{\mathfrak{p}R_{\mathfrak{p}}}}\} =$$

since $R_{\mathfrak{p}}$ is a noetherian ring and the Gabriel filter $\mathcal{L}(\sigma_{\mathfrak{p}R_{\mathfrak{p}}})$ has a filter basis of the form $\{\mathfrak{p}^n R_{\mathfrak{p}}\colon n \in \mathbb{N}\}$, so

$$= \varprojlim\{R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}}\colon n \in \mathbb{N}\} = \widehat{R_{\mathfrak{p}}}. \quad ∎$$

Similar statements may be established for σ-finitely generated modules, as we will see:

COROLLARY 1.3: *Let M be a σ-finitely generated R-module. Then*

$$M^{(\sigma,\pi_{\mathfrak{p}})} \cong \widehat{M_{\mathfrak{p}}}.$$

Proof: As $\sigma \leq \sigma_{R\setminus\mathfrak{p}}$, it follows that $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module. As before, we have

$$M^{(\sigma,\pi_{\mathfrak{p}})} \cong (M_{\mathfrak{p}})^{(\sigma,\pi_{\mathfrak{p}})}$$

and the R-submodules N of $M_{\mathfrak{p}}$ such that $M_{\mathfrak{p}}/N \in \mathcal{F}_\sigma \cap \mathcal{T}_{\pi_{\mathfrak{p}}}$ are exactly the $R_{\mathfrak{p}}$-submodules of $M_{\mathfrak{p}}$ such that the quotient belong to $\mathcal{T}_{\sigma_{\mathfrak{p}R_{\mathfrak{p}}}}$. Finally, the topology of $M_{\mathfrak{p}}$ induced by $\sigma_{\mathfrak{p}R_{\mathfrak{p}}}$ has a filter basis of the form $\{\mathfrak{p}^n M_{\mathfrak{p}}\colon n \in \mathbb{N}\}$. Combining the previous statements proves the assertion. ∎

Remark 1.4: It is well known that if $M_{\mathfrak{p}}$ is finitely generated as an $R_{\mathfrak{p}}$-module, then $\widehat{M_{\mathfrak{p}}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widehat{R_{\mathfrak{p}}}$, cf. [1, Théorème III.3.3]. Also, if $\mathfrak{p} \notin \mathrm{Supp}(M)$, then $M_{\mathfrak{p}} = 0$ and so $M^{(\sigma,\pi_{\mathfrak{p}})} = 0$.

2. (σ, σ^1)-completion

We proceed with studying the first skeleton σ^1 of the torsion theory σ in $R-\mathrm{mod}$ using the set $\mathcal{C}(\sigma)$ of those prime ideals which are maximal in $\mathcal{K}(\sigma)$

PROPOSITION 2.1: *Let σ be a torsion theory in $R-\mathrm{mod}$ and consider the torsion theory $\bigvee\{\pi_{\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\sigma)\}$. Then $\sigma^1 = \bigvee\{\pi_{\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\sigma)\}$.*

Proof: Since $\sigma \leq \vee\{\pi_{\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\sigma)\}$ and $\sigma \leq \sigma^1$, then σ^1 and $\vee\{\pi_{\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\sigma)\}$ are determined by $\mathcal{K}(\sigma^1)$ and $\mathcal{K}(\vee\{\pi_{\mathfrak{p}}: \mathfrak{p} \in \mathcal{C}(\sigma)\})$ respectively. Thus by definition we have $\mathcal{K}(\pi_{\mathfrak{p}}) = \mathcal{K}(\sigma) \setminus \{\mathfrak{p}\}$, and so from the next chain of equalities it follows the assertion.

$$\mathcal{K}(\vee\pi_{\mathfrak{p}}) = \cap\mathcal{K}(\pi_{\mathfrak{p}}) = \cap(\mathcal{K}(\sigma) \setminus \{\mathfrak{p}\}) = \mathcal{K}(\sigma) \setminus \mathcal{C}(\sigma) = \mathcal{K}(\sigma^1). \qquad \blacksquare$$

COROLLARY 2.2: *If M is an R-module, then*

$$M^{(\sigma,\sigma^1)} = M^{(\sigma,\vee\pi_{\mathfrak{p}})}.$$

With this identification between σ^1 and $\vee\pi_{\mathfrak{p}}$, we can handle the torsion theory σ^1, in an easy way, by using the description of the Gabriel filter of the join of a family of torsion theories given in [16, (1.8)]. Hence we have:

$$\mathcal{L}(\sigma^1) = \mathcal{L}(\vee\pi_{\mathfrak{p}}) = \{I_{\mathfrak{p}_1} I_{\mathfrak{p}_2} \ldots I_{\mathfrak{p}_n}: I_{\mathfrak{p}_i} \in \mathcal{L}(\pi_{\mathfrak{p}_i})\}.$$

The following definition should give a method to decompose modules, and will be useful for our purposes:

Definition 2.3: Let R be a ring, two ideals I and J of R will be called σ-**comaximal** if $I + J \in \mathcal{L}(\sigma)$. A finite set of ideals will be called **pairwise** σ-**comaximal** if all couples of them are σ-comaximal.

An immediate consequence of the definition is that I and J are σ-comaximal if, and only if, $Q_\sigma(I + J) = Q_\sigma(R)$.

PROPOSITION 2.4:

1. *I, J are σ-comaximal ideals of R if, and only if, the natural homomorphism*

$$Q_\sigma(R/I\cap J)\to Q_\sigma(R/I)\times Q_\sigma(R/J)$$

is an isomorphism.

2. *If I, I_1, ..., I_n are pairwise σ-comaximal ideals of R, then $I+I_1\dots I_n\in\mathcal{L}(\sigma)$.*
3. *If I_1, ..., I_n are pairwise σ-comaximal ideals of R, then*

$$(I_1\cap\dots\cap I_n)/I_1\cdots I_n\in\mathcal{L}(\sigma).$$

4. *If I_1, ..., I_n are pairwise σ-comaximal ideals of R, then the natural homomorphism*

$$Q_\sigma(R/I_1\cap\dots\cap I_n)\to Q_\sigma(R/I_1)\times\dots\times Q_\sigma(R/I_n)$$

is an isomorphism.

Proof of 1: Suppose I, J are σ-comaximal ideals of R. The natural homomorphism

$$\Phi\colon R\to R/I\times R/J$$

induces a homomorphism $Q_\sigma(\Phi)$ with factorization

$$Q_\sigma(\Phi)\colon Q_\sigma(R)\to Q_\sigma(R/I\cap J)\overset{Q_\sigma(\Phi')}{\to}Q_\sigma(R/I\times R/J).$$

We will prove that $(R/I\times R/J)/\Phi(R)\in\mathcal{T}_\sigma$, then since $Q_\sigma(\Phi')$ is a monomorphism, it will be an isomorphism. This will be enough to obtain the assertion because σ is of finite type, and so Q_σ commutes with direct sums.

We claim that $(R/I\times R/J)/\Phi(R)$ is σ-torsion if, and only if, I and J are σ-comaximal ideals of R.

First we remark that $(I+J)/I\times(I+J)/J\subseteq\Phi(R)$; let $(j+I,i+J)\in(I+J)/I\times(I+J)/J$ with $j\in J$ and $i\in I$, thus if we define $r=i+j$, it is easy to see that $\Phi(r)=(j+I,i+J)$, which proves the remark.

Let us assume that I and J are σ-comaximal ideals of R and let $(a+I,b+J)\in R/I\times R/J$, there exists $H\in\mathcal{L}(\sigma)$ such that Ha, $Hb\subseteq I+J$, and hence $H(a+I,b+J)\subseteq(I+J)/I\times(I+J)/J\subseteq\Phi(R)$.

For the converse, if $(R/I\times R/J)/\Phi(R)$ is σ-torsion and $r\in R$, then $(r+I,0+J)\in(R/I\times R/J)$ and there exists $H\in\mathcal{L}(\sigma)$ such that $H(r+I,0+J)\subseteq\Phi(R)$.

From this inclusion we have that for any $h \in H$ there exists $r_h \in R$ such that $r_h + I = hr + I$ and $r_h + J = 0 + J$, so $hr \in I + J$ and $Hr \subseteq I + J$ and hence I and J are σ-comaximal ideals of R.

Proof of 2: Since $I + I_i \in \mathcal{L}(\sigma)$ for $i = 1, \ldots, n$, we have $\Pi_{i=1}^{n}(I + I_i) \in \mathcal{L}(\sigma)$ and it is clear the next inclusion

$$\Pi_{i=1}^{n}(I + I_i) \subseteq I + I_1 \ldots I_n,$$

hence $I + I_1 \ldots I_n \in \mathcal{L}(\sigma)$.

Proof of 3: In this point, using the above assertions, we have that $(I_i + \Pi_{j>i} I_j) \in \mathcal{L}(\sigma)$ for $i = 1, \ldots, n-1$. The product of all of these ideals is contained in the sum $\sum_{i=1}^{n-1} I_1 \ldots I_{i-1} I_{i+1} \ldots I_n$, so the next inclusion is immediate

$$\left(\prod_{i=1}^{n-1}(I_i + \Pi_{j>i} I_j)\right)(I_1 \cap \ldots \cap I_n) \subseteq I_1 \ldots I_n$$

whence follows the assertion.

Proof of 4: We consider the canonical map

$$\Phi: R \to R/I_1 \times \ldots \times R/I_n.$$

As in statement 1, we only need to show that $(R/I_1 \times \ldots \times R/I_n)/\Phi(R)$ is σ-torsion. Let $(x_1 + I_1, \ldots, x_n + I_n) \in R/I_1 \times \ldots \times R/I_n$. Since $I_i + I_1 \ldots I_{i-1} I_{i+1} \ldots I_n$ is in $\mathcal{L}(\sigma)$ for all $i = 1, \ldots, n$, then there exist $H_i \in \mathcal{L}(\sigma)$ such that $H_i x_i \subseteq I_i + I_1 \ldots I_{i-1} I_{i+1} \ldots I_n$ for each $i = 1, \ldots, n$. If we define $H = H_1 \cap \ldots \cap H_n$, then for any $h \in H$ there exist $y_i \in I_i$ and $x_{(i)} \in I_1 \ldots I_{i-1} I_{i+1} \ldots I_n$ such that $hx_i = y_i + x_{(i)}$. If we take $x = x_{(1)} + \ldots + x_{(n)}$, then

$$x + I_i = x_{(i)} + I_i = hx_i + I_i.$$

So $H(x_1 + I_1, \ldots, x_n + I_n) \subseteq \Phi(R)$ and this proves the assertion. ∎

LEMMA 2.5: *Let $I_1, \ldots, I_n$ be pairwise σ-comaximal ideals of R and let M be an R-module, then the quotient*

$$(I_1 M \cap \cdots \cap I_n M)/(I_1 \cdots I_n)M$$

is σ-torsion.

Proof: From Proposition 2.4 we obtain that $\Pi_{i=1}^{n-1}(I_i + \Pi_{j>i} I_j) \in \mathcal{L}(\sigma)$. Let $m \in I_1 M \cap \ldots \cap I_n M$, then

$$(\prod_{i=1}^{n-1}(I_i + \prod_{j>i} I_j))m \subseteq (I_1 \cdots I_n)M$$

and hence we have the assertion. ∎

The last statement in the Proposition 2.4 generalizes to arbitrary R-modules M. More precisely, we can enunciate:

PROPOSITION 2.6: *Let $I_1, \ldots, I_n$ be pairwise σ-comaximal ideals of R and let M be an R-module, then the natural homomorphism*

$$Q_\sigma(M/I_1 \cdots I_n M) \to Q_\sigma(M/I_1 M) \times \cdots \times Q_\sigma(M/I_n M)$$

is an isomorphism.

PROPOSITION 2.7:

1. *Let I and J be ideals of R such that $R/I \in \mathcal{F}_\sigma \cap \mathcal{T}_{\pi_\mathfrak{p}}$ and $R/J \in \mathcal{F}_\sigma \cap \mathcal{T}_{\pi_\mathfrak{q}}$ with $\mathfrak{p} \neq \mathfrak{q} \in \mathcal{C}(\sigma)$. Then I, J are σ-comaximal.*
2. *Let M be a σ-finitely generated R-module, and I, J as before, then*

$$\mathrm{Hom}_R\left(Q_\sigma\left(M/IM\right), Q_\sigma\left(M/JM\right)\right) = 0.$$

Proof of 1: Let $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R)\colon I \subseteq \mathfrak{p}\}$. We know that $V(I+J) = V(I) \cap V(J)$. Now, observe that $\varnothing \neq \mathrm{Ass}(R/I) = \{\mathfrak{p}\}$ and also that $\mathfrak{p}$ is just the minimal prime ideal of $V(I)$. The same is true for J. Therefore $V(I+J) \subseteq \mathcal{Z}(\sigma)$ and hence $R/(I+J) \in \mathcal{T}(\sigma)$.

Proof of 2: The module $Q_\sigma(M/JM)$ belong to $\mathcal{F}_\sigma \cap \mathcal{T}_{\pi_\mathfrak{p}} \subseteq \mathcal{F}_{\sigma_{R\setminus\mathfrak{q}}}$. So, we only have to prove that $Q_\sigma(M/IM)$ is $\sigma_{R\setminus\mathfrak{q}}$-torsion to obtain the assertion. It is clear that M/IM is $\sigma_{R\setminus\mathfrak{q}}$-torsion because $\varnothing \neq \mathrm{Ass}(M/IM) = \{\mathfrak{p}\} \subseteq \mathcal{Z}(\sigma_{R\setminus\mathfrak{q}})$. Now, as $\sigma_{R\setminus\mathfrak{q}}$ is stable (since $\sigma \leq \sigma_{R\setminus\mathfrak{q}}$) hence $Q_\sigma(M/IM)$ is $\sigma_{R\setminus\mathfrak{q}}$-torsion. ∎

LEMMA 2.8: *Let M be a σ-finitely generated R-module, then*

$$M^{(\sigma,\sigma^1)} \cong \varprojlim\{M^{(\sigma,\pi_\mathfrak{p})}\colon \mathfrak{p} \in \mathcal{C}(\sigma)\}.$$

In particular, this holds for $M = R$.

Proof: The proof will be given for R but actually works for any σ-finitely generated R-module M.

$$R^{(\sigma,\sigma^1)} = R^{(\sigma,\vee\pi_{\mathfrak{p}})} \cong \varprojlim\{Q_\sigma(R/I_{\mathfrak{p}_1}\dots I_{\mathfrak{p}_n})\colon I_{\mathfrak{p}_i} \in \mathcal{L}(\mathfrak{p}_i),\ \mathfrak{p}_i \in \mathcal{C}(\sigma)\}.$$

We observe that $I_{\mathfrak{p}_i}$ and $I_{\mathfrak{p}_j}$ are σ-comaximal when $\mathfrak{p}_i \neq \mathfrak{p}_j$. By Proposition 2.4 there exists an isomorphism

$$Q_\sigma(R/I_{\mathfrak{p}_1}\dots I_{\mathfrak{p}_n}) \cong Q_\sigma(R/I_{\mathfrak{p}_1}) \times \dots \times Q_\sigma(R/I_{\mathfrak{p}_n}).$$

Now we use that $\mathrm{Hom}_R(Q_\sigma(R/I_{\mathfrak{p}_i}), Q_\sigma(R/I_{\mathfrak{p}_j})) = 0$ when $\mathfrak{p}_i \neq \mathfrak{p}_j$. Therefore, the inverse limit yields

$$R^{(\sigma,\sigma^1)} \cong \varprojlim\{\varprojlim\{Q_\sigma(R/I_{\mathfrak{p}_i})\colon R/I_{\mathfrak{p}_i} \in \mathcal{F}_\sigma \cap \mathcal{T}_{\pi_{\mathfrak{p}_i}}\}\colon \mathfrak{p}_i \in \mathcal{C}(\sigma)\} \cong$$
$$\cong \varprojlim\{R^{(\sigma,\pi_{\mathfrak{p}})}\colon \mathfrak{p} \in \mathcal{C}(\sigma)\}. \qquad ∎$$

THEOREM 2.9: *If M is σ-finitely generated, then*

$$M^{(\sigma,\sigma^1)} \cong \Pi\{\widehat{M_{\mathfrak{p}}}\colon \mathfrak{p} \in \mathcal{C}(\sigma)\}.$$

In particular, this holds for $M = R$.

Proof: From the above lemma

$$M^{(\sigma,\sigma^1)} \cong \varprojlim\{M^{(\sigma,\pi_{\mathfrak{p}})}\colon \mathfrak{p} \in \mathcal{C}(\sigma)\}$$

and using the calculations at the beginning of the Section 1 we have:

$$M^{(\sigma,\sigma^1)} \cong \varprojlim\{\widehat{M_{\mathfrak{p}}}\colon \mathfrak{p} \in \mathcal{C}(\sigma)\}.$$

In order to show that this inverse limit is a product, we will prove that there does not exist any non zero homomorphism between $\widehat{M_{\mathfrak{p}}}$ and $\widehat{M_{\mathfrak{q}}}$ when $\mathfrak{p} \neq \mathfrak{q}$. Let $f\colon \widehat{M_{\mathfrak{p}}} \to \widehat{M_{\mathfrak{q}}}$ be a homomorphism, then f is zero if, and only if, for any canonical projection $p_n\colon \widehat{M_{\mathfrak{q}}} \to M_{\mathfrak{q}}/\mathfrak{q}^n M_{\mathfrak{q}}$, the composition $p_n f$ is zero. Thus to prove that f is always zero, since $M_{\mathfrak{q}}/\mathfrak{q}^n M_{\mathfrak{q}} \in \mathcal{F}_{\sigma_{R\setminus\mathfrak{q}}}$, we only need to prove that $\widehat{M_{\mathfrak{p}}}$ is $\sigma_{R\setminus\mathfrak{q}}$-torsion. By Remark 1.4 we have that $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widehat{R_{\mathfrak{p}}} \cong \widehat{M_{\mathfrak{p}}}$, hence it is enough to prove that $\widehat{R_{\mathfrak{p}}}$ is $\sigma_{R\setminus\mathfrak{q}}$-torsion. But $R/\mathfrak{p}$ is $\sigma_{R\setminus\mathfrak{q}}$-torsion, and since $\sigma_{R\setminus\mathfrak{q}}$ is stable, also is $E(R/\mathfrak{p})$. On the other hand, we know that $\widehat{R_{\mathfrak{p}}} \cong \mathrm{End}(E(R/\mathfrak{p}))$, [11, Corollary 3.10]. Let $\phi \in \mathrm{End}(E(R/\mathfrak{p}))$; ϕ is determined by $\phi(1) \in E(R/\mathfrak{p}) \in \mathcal{T}_{\sigma_{R\setminus\mathfrak{q}}}$. So there exists $I \in \mathcal{L}(\sigma_{R\setminus\mathfrak{q}})$ such that $I\phi(1) = 0$ and therefore $I\phi = 0$. Thus, $\phi \in \sigma_{R\setminus\mathfrak{q}}\left(\widehat{R_{\mathfrak{p}}}\right)$ and the proof is finished. ∎

COROLLARY 2.10: *Under the same assumptions as before,*

$$M^{(\sigma,\sigma^1)} = \Pi\{\widehat{M_\mathfrak{p}}\colon \mathfrak{p} \in \mathcal{C}(\sigma) \cap \mathrm{Supp}(M)\}.$$

Proof: It follows from Remark 1.4 and the previous Theorem. ∎

3. Exactness of the (σ, σ^1)-completion functor.

We had obtained in [3] the left exactness of the (σ, τ)-completion functor. But there are some cases in which the (σ, σ^1)-completion is an exact functor on σ-finitely generated R-modules, as the next Proposition shows.

PROPOSITION 3.1: *Every exact sequence of σ-finitely generated modules*

$$0 \to M' \to M \to M'' \to 0$$

induces an exact sequence

$$0 \to (M')^{(\sigma,\sigma^1)} \to M^{(\sigma,\sigma^1)} \to (M'')^{(\sigma,\sigma^1)} \to 0.$$

Proof: The localization at $\mathfrak{p}$ is always exact, thus for each $\mathfrak{p} \in \mathcal{C}(\sigma)$ we find an exact sequence:

$$0 \to (M')_\mathfrak{p} \to M_\mathfrak{p} \to (M'')_\mathfrak{p} \to 0.$$

As for all $\mathfrak{p} \in \mathcal{C}(\sigma)$ we have $\sigma \leq \sigma_{R \setminus \mathfrak{p}}$ and each module in the above sequence is a finitely generated $R_\mathfrak{p}$-module. Now, making use of the results about classical completion, we obtain an exact sequence

$$0 \to \widehat{(M')_\mathfrak{p}} \to \widehat{M_\mathfrak{p}} \to \widehat{(M'')_\mathfrak{p}} \to 0.$$

From this and Theorem 2.9 we complete the proof. ∎

COROLLARY 3.2: *For every pair of σ-finitely generated R-modules M and N, we have*

$$M^{(\sigma,\sigma^1)} \oplus N^{(\sigma,\sigma^1)} \cong (M \oplus N)^{(\sigma,\sigma^1)}.$$

As it is well known in the I-adic completion case there exists an isomorphism between the completion of a finitely generated module and the tensor product with the completion $\widehat{R}$ of R. The same result is not proved at the moment for this general completion, but we will approach to it as far as possible.

First of all, we define a homomorphism $\mu_M\colon R^{(\sigma,\sigma^1)} \otimes M \to M^{(\sigma,\sigma^1)}$ as follows: $\mu_M((r_I)_{I\in\mathcal{L}(\sigma^1)} \otimes m) = (k_N)_{N\in\mathcal{L}_{\sigma^1}(M)}$, determined by the property that

$$k_{IM} = r_I\, m_{IM} \text{ when } N = IM.$$

(Here we represent by m_{IM} the image of $m + IM \in M/IM$ in $Q_\sigma(M/IM)$).

Since $R^{(\sigma,\sigma^1)} \otimes_R R \cong R^{(\sigma,\sigma^1)}$, using Corollary 3.2, we obtain

$$R^{(\sigma,\sigma^1)} \otimes_R R^n \cong (R^n)^{(\sigma,\sigma^1)}.$$

PROPOSITION 3.3: *Let M be a σ-finitely generated R-module. Choose $N \le M$ finitely generated such that $M/N \in \mathcal{T}_\sigma$. Then*

$$R^{(\sigma,\sigma^1)} \otimes_R N \cong N^{(\sigma,\sigma^1)}$$

and in the following diagram,

$$\begin{array}{ccc} R^{(\sigma,\sigma^1)} \otimes_R N & \longrightarrow & R^{(\sigma,\sigma^1)} \otimes_R M \\ \cong\big\downarrow & & \big\downarrow \\ N^{(\sigma,\sigma^1)} & \xrightarrow{\ \cong\ } & M^{(\sigma,\sigma^1)} \end{array}$$

the left-vertical and the bottom-horizontal arrows are isomorphims of R-modules.

Proof: Let N be the adequate submodule of M; it is possible to find an exact sequence

$$0 \to K \to R^n \to N \to 0$$

with K σ-finitely generated module and $n \in \mathbb{N}$. Since the next diagram with exact rows

$$\begin{array}{ccccccccc} & & R^{(\sigma,\sigma^1)} \otimes_R K & \longrightarrow & R^{(\sigma,\sigma^1)} \otimes_R R^n & \longrightarrow & R^{(\sigma,\sigma^1)} \otimes_R N & \longrightarrow & 0 \\ & & \mu_K\big\downarrow & & \mu_{R^n}\big\downarrow & & \mu_N\big\downarrow & & \\ 0 & \longrightarrow & K^{(\sigma,\sigma^1)} & \longrightarrow & (R^n)^{(\sigma,\sigma^1)} & \longrightarrow & N^{(\sigma,\sigma^1)} & \longrightarrow & 0 \end{array}$$

is commutative and μ_{R^n} is an isomorphism, then μ_N is an epimorphism. To get information about μ_K we consider the following commutative diagram

$$\begin{array}{ccccccc} R^{(\sigma,\sigma^1)} \otimes_R H & \longrightarrow & R^{(\sigma,\sigma^1)} \otimes_R K & \longrightarrow & R^{(\sigma,\sigma^1)} \otimes_R K/H & \longrightarrow & 0 \\ \mu_H\big\downarrow & & \mu_K\big\downarrow & & 0\big\downarrow & & \\ H^{(\sigma,\sigma^1)} & \xrightarrow{\ \cong\ } & K^{(\sigma,\sigma^1)} & \longrightarrow & 0 & & \end{array}$$

where H is a finitely generated submodule of K such that $K/H \in T_\sigma$. The latter diagram shows that μ_K is an epimorphism because it is the second in a composition which is epimorphism. Now, turning to the former diagram, an easy diagram chase gives us that μ_N is an isomorphism. ∎

This Proposition can be particularized to perfect torsion theories, in this case we obtain a more efficient description of the completion.

COROLLARY 3.4: *When σ is a perfect torsion theory (i. e.: Q_σ is a exact functor on $R-\mathrm{mod}$), for any σ-finitely generated R-module M we have:*

$$R^{(\sigma,\sigma^1)} \otimes_R M \cong M^{(\sigma,\sigma^1)}.$$

Proof: In the diagram in the above Proposition

$$\begin{array}{ccc} R^{(\sigma,\sigma^1)} \otimes_R N & \longrightarrow & R^{(\sigma,\sigma^1)} \otimes_R M \\ \cong\downarrow & & \downarrow \\ N^{(\sigma,\sigma^1)} & \xrightarrow{\cong} & M^{(\sigma,\sigma^1)} \end{array}$$

it is clear that the top-horizontal arrow is a monomorphism. We claim that it is an isomorphism and this shows the assertion. Consider the exact sequence

$$0 \to R^{(\sigma,\sigma^1)} \otimes_R N \to R^{(\sigma,\sigma^1)} \otimes_R M \to R^{(\sigma,\sigma^1)} \otimes_R M/N$$

if we compute the latter module, we have

$$R^{(\sigma,\sigma^1)} \otimes_R M/N \cong R^{(\sigma,\sigma^1)} \otimes_R Q_\sigma(R) \otimes_R M/N \cong R^{(\sigma,\sigma^1)} \otimes_R Q_\sigma(M/N)$$

where we used the σ-injectivity of $R^{(\sigma,\sigma^1)}$ and the exactness of Q_σ. Now, since M/N is σ-torsion, $R^{(\sigma,\sigma^1)} \otimes_R Q_\sigma(M/N)$ is zero. Hence we have an isomorphism

$$R^{(\sigma,\sigma^1)} \otimes_R N \cong R^{(\sigma,\sigma^1)} \otimes_R M,$$

and as consequence we obtain the assertion. ∎

As a subproduct of this result we have that if σ is a perfect torsion theory, then $R^{(\sigma,\sigma^1)}$, the (σ,σ^1)-completion of R, is a flat R-module.

Example 3.5: Let R be a Dedekind domain. If I is a non zero ideal of R, then the I-adic completion of R can be obtain as the (σ,τ)-completion when we consider σ the trivial torsion theory and $\tau = \sigma_I$. It is clear that in this case $\mathcal{Z}(\sigma^1) =$

$\mathrm{Spec}(R)\setminus\{0\}$ and since $\mathcal{Z}(\sigma_I) = V(I) \subseteq \mathrm{Spec}(R)\setminus\{0\}$ hence $\sigma \leq \tau \leq \sigma^1$. In this situation applying the above results we have

$$R^{(\sigma,\tau)} = \Pi\{\widehat{R_\mathfrak{p}}: \mathfrak{p} \in V(I)\}.$$

In this particular case the index family is finite and then the (σ, τ)-completion has the correspondent universal property.

Example 3.6: Let R be a Krull domain. Consider σ the torsion theory defined by $\mathcal{K}(\sigma) = \mathrm{Spec}^{(1)}(R)$. Thus, we are in condition to apply the above results. As before, σ^1 is the usual torsion theory in a domain. To compute $R^{(\sigma,\sigma^1)}$ we observe that for every $\mathfrak{p} \in \mathrm{Spec}^{(1)}(R)\setminus\{0\}$ the ring of quotients $R_\mathfrak{p}$ is a local principal ideal domain, and so $\mathfrak{p}R_\mathfrak{p}$ is generated by a single element, said $a_\mathfrak{p}$. For these domains the classical completion can be described in terms of a power series ring in an indeterminate, said $X_\mathfrak{p}$, as follow

$$\widehat{R_\mathfrak{p}} \cong R_\mathfrak{p}[[X_\mathfrak{p}]]/(X_\mathfrak{p} - a_\mathfrak{p}),$$

see [7, Proposition 3.3.]. By Theorem 2.9 we obtain an isomorphism decomposing the (σ, σ^1)-completion.

$$R^{(\sigma,\sigma^1)} \cong \Pi\{\widehat{R_\mathfrak{p}}: \mathfrak{p} \in \mathrm{Spec}^{(1)}\setminus\{0\}\} = \Pi\{R_\mathfrak{p}[[X_\mathfrak{p}]]/(X_\mathfrak{p} - a_\mathfrak{p}): \mathfrak{p} \in \mathrm{Spec}^{(1)}\setminus\{0\}\}.$$

References

[1] N. Bourbaki, *Algèbre Commutative*, Masson, Paris, 1985.

[2] W. Brandal, *Completions of commutative topological rings*, Comm. Algebra **20** (1992), 3381–3391.

[3] J. L. Bueso, P. Jara and E. Santos, *Completion*, to appear in Communications in Algebra.

[4] J. L. Bueso, P. Jara and A. Verschoren, *Duality, localization and completion*, to appear in Journal of Pure and Applied Algebra.

[5] J. L. Bueso, B. Torrecillas and A. Verschoren, *Local Cohomology and Localization*, Pitman Research Notes in Mathematical Series No. 226, Longman Scientific and Technical, Harlow, 1990.

[6] J. S. Golan, *Torsion Theories*, Pitman Monograph and Surveys in Pure and Applied Mathematics No. 29, Longman Scientific and Technical, Essex, 1986.

[7] S. Greco and P. Salmon, *Topics in $\mathfrak{m}$-adic Topologies*, Springer-Verlag, Berlin, 1971.

[8] J. Lambek, *Localization and completion*, J. Pure and Applied Algebra **2** (1972), 343–370.

[9] E. Matlis, *Torsion-free Modules*, The Chicago University Press, Chicago, 1972.

[10] C. Menini and A. Orsatti, *Duality over a quasi-injective module and commutative $\mathcal{F}$-reflexive rings*, Symposia Mathematica **XXIII** (1979), 145–179, Academic Press.

[11] C. Nastasescu, *La structure des modules par rapport à une topologie additive*, Tohôhu Math. J. **22** (1974), 173–201.

[12] J. Rios Montes, *Algunos funtores relacionados con la completación de módulos respecto a un filtro de Gabriel*, Rev. Mat. Hisp.-Amer. **42** (1982), 201–219.

[13] D. W. Sharpe and P. Vamos, *Injective Modules*, Cambridge University Press, Cambridge, 1972.

[14] B. Stenström, *On the completion of modules in an additive topology*, J. Algebra **16** (1970), 523–540.

[15] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.

[16] A. Verschoren, *Compatibility and stability*, Notas de Matem tica, Vol. 3, Universidad de Murcia, 1990.